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The topology of the zero-level lines of the components of the acceleration vector in subsonic flows☆

A.I. Rylov

Novosibirsk

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Abstract

The steady subsonic flow past bodies of finite dimensions, when the stream is unbounded and uniform at infinity is considered. The structure formed by the stationary points (points where both components of the acceleration vector vanishes), by the zero-level of the components of the acceleration vector emerging from them and the body past which the flow occurs is studied. It is shown that each of the above-mentioned lines must reach the surface of the body past which the flow takes place. This fact, in particular, enables one to estimate the overall number of streamlines with zero curvature emerging from the stationary points in terms of the number of zeros of the curvature of the streamlines on the body around which the flow takes place, including the branch points of a dividing streamline. With a view to refining the above mentioned number of zeros, the known solution for the neighbourhoods of the branch points of a streamline is considered and the singularity of the flow in the neighbourhoods of points of discontinuity of the curvature of the wall around which the flow occurs is investigated. In order to illustrate the above, certain properties of the flow past convex bodies are refined and a fairly broad class of so-called convex-concave bodies with zero angle of tapering of the trailing edge is constructed and considered. It is shown that, for this body, there are not more than four zeros of the curvature of the streamline and, as a consequence, there are no branch points of the isobars and isoclines in the flow field, including at infinity, an infinitely distant point is the sole stationary point and, most important of all, in the case of the flow past the given bodies the values of the circulation and the lifting force cannot vanish. The mathematical apparatus employed is based on the equations of gas dynamics constructed earlier for certain combinations of the components of the acceleration vector.

1. Zero-level lines of the components of the acceleration vector

Consider a plane potential subsonic flow past a body of a perfect gas (inviscid and non-heat-conducting), when the stream is unbounded and horizontal at infinity. The flow is characterized by the existence of a dividing streamline with two branching points t and d lying on the body. The dividing streamline reaches the body from infinity on the left at the leading branch point t, where it divides into two branches, each of which is adjacent to the body past which the flow occurs. These branches are again combined into a single dividing streamline at the trailing branching point d which departs on the right to infinity. There is no more than one sharp edge with an angle $0 \le \sigma < \pi/2$ on the body and, according to the Zhukovskii condition, when such an edge exists it is the trailing branch point d, while the leading branch point t is in any case also a stagnation point. There are no other branch points, apart from t and d, in the flow being considered and, consequently, there are also no stagnation points outside the body in the flow.

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E-mail address: rylov@math.nsc.ru.

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In the plane of the potential (φ, ψ), the flow being investigated is described by the equations[1,2]

$$kz_\varphi + \theta_\psi = 0, \quad z_\psi - \theta_\varphi = 0 \quad \left(z = \int \frac{\rho dq}{q}, \ k = \frac{1-M^2}{\rho^2}\right) \tag{1.1}$$

Henceforth, φ and ψ are the potential and the stream function, ρ is the density, q and θ are the modulus and angle of inclination of the velocity vector and M is the Mach number.

The principal geometrical objects of the investigation are the zero-level lines of the longitudinal component (along a streamline) F and the transverse component (along the left normal to the stream line) G of the acceleration vector and the stationary points $F = G = 0$. The functions F and G are related to the derivatives from system (1.1) as follows:

$$F = z_\varphi q^3 \rho^{-1}, \quad G = \theta_\varphi q^3$$

by which the choice of system (1.1) as the initial system is largely determined. Note also that the function G is identical, apart from a positive factor, with the curvature of a streamline.

To study the lines $F = 0$ and $G = 0$, the equations[3]

$$kU_\varphi - V_\psi = 0, \quad U_\psi + V_\varphi = 0 \tag{1.2}$$

are most suitable, with the new independent variables U and V:

$$U = \frac{z_\varphi}{kz_\varphi^2 + \theta_\varphi^2}, \quad V = \frac{\theta_\varphi}{kz_\varphi^2 + \theta_\varphi^2} \tag{1.3}$$

We recall that system (1.2) is obtained by differentiating system (1.1) with respect to φ and subsequent transformation of the resulting inhomogeneous system into a homogeneous system using an algorithm[4] based on the use of two-parameter solutions. In the case under consideration, a helical flow, which is the superposition of a flow from a source and a flow of the potential vortex type,[5,6] will be such a flow. However, the relation between system (1.2) and helical flow already follows from the fact that the obvious solution $U = C_1 = \text{const}$, $V = C_2 = \text{const}$ also exactly gives a helical flow.

Note also that the compatibility conditions of system (1.2) in the supersonic zone can be written in the form of the so-called transport equations.[2,3]

When $M < 1$, the homogeneous system (1.2) is elliptic and, therefore,[4] the functions U and V in the flow being considered possess the property of monotonicity: each of these functions is monotonic along a level line of the other function.

Here, it is opportune to define the level line more precisely.

1.1. Definition

We mean by a level line of the function U (the function V) a line $U = \text{const}$ ($V = \text{const}$) for the continuation of which, to be specific, the left-hand branch is chosen on passing a branch point.

In order to demonstrate the monotonicity properties of the functions U and V, we will write expressions for the derivatives U_l and V_l which are calculated along the level lines of the functions V and U respectively[3]:

$$\begin{aligned} &V = \text{const}, \quad U_l = V_n \frac{\rho(1 - M^2 \sin^2\chi)}{1 - M^2} \\ &U = \text{const}, \quad V_l = -U_n \frac{1 - M^2 \sin^2\delta}{\rho} \end{aligned} \tag{1.4}$$

where χ and δ are the angles made by the level lines with the velocity vector.

The derivatives V_n and U_n, calculated along the left-hand normals to the lines $V = \text{const}$ and $U = \text{const}$ respectively in the domain of ellipticity can only vanish at isolated points. Otherwise, for example when the equality $V_n = 0$ is satisfied along a certain segment of a line $V = \text{const}$, all four derivatives $U_\varphi, \ldots, V_\psi$ must, in fact, be equal to zero, which is only possible if there is a helical flow in the whole domain. It follows from what has been said, from the definition of a level line and from relations (1.4), that, along a level line $V = \text{const}$ ($U = \text{const}$), the derivative U_l (the derivative V_l) does not change its sign. However, it should be emphasized that, unlike isobars, isoclines and other level lines, the analysis of which is based on the method of level lines,[4,7–11] an infinite set of closed level lines $U = \text{const} \neq 0$ and $V = \text{const} \neq 0$ exists in the subsonic flow domain being considered, when there are stationary points $F = G = 0$, which pass through the above mentioned points (these are the points of indeterminacy of the functions U and V). In the case of a circular bypass of a stationary point, the functions U and V take all values in the range from $-\infty$ to $+\infty$. This fact will be demonstrated in greater detail below when considering stationary points. Note that the analysis of the lines $U = \text{const}$ and $V = \text{const}$ presented earlier in Refs. 3,11 was solely restricted by the domains of unique definition of the functions U and V.

The lines $U = F = 0$ and $V = G = 0$ occupy a special place among the set of level lines of the functions U and V. This is associated, first of all, with the fact that no combination of components of the acceleration vector is monotonic along each of these lines but only one of the components taken with a certain constant sign factor. The physical meaning of these lines is quite clear and, what is more, in experiments in which the streamlines are visualized, the line $V = G = 0$ is readily overlooked as the geometrical locus of the points of inflection of the streamlines.

It is found that the analysis of the zero-level lines of the components of the acceleration vector is most effective if the stationary points $F = G = 0$ are the initial points of the branches of the above-mentioned lines. We will therefore now consider some fairly obvious properties of the above mentioned points.

2. Stationary points

In the subsonic flow being considered, there are two types of stationary points outside the boundaries of the body in the flow. These are the branch points of the isobars and isoclines, which are located at a finite distance from the body and at an infinitely distant point.

2.1. Branch point of isobars and isoclines

This point has been discussed in considerable detail (see Ref. 5 and other textbooks). Here, first of all, the dependence of the number of lines $F = 0$ and $G = 0$ emerging from a branch point on the number of isobars and isoclines emerging from this same point is of interest.

We shall make use of the method of "frozen" parameters for which, in an infinitelesimal neighbourhood of the branch point being investigated in which a zero subscript is assigned to the parameters, we transform the initial system (1.1) to a system of the Cauchy-Riemann type

$$S_\varphi + T_\phi = 0, \quad S_\phi - T_\varphi = 0 \tag{2.1}$$

where

$$S = \sqrt{k_0}(z - z_0), \quad T = \theta - \theta_0, \quad \phi = \sqrt{k_0}\psi$$

The periodic solution at the branch point in the local system of coordinates (R, ω) in the (φ, ψ) plane of interest has the form

$$S = AR^n\cos(n\omega - t_*), \quad T = AR^n\sin(n\omega - t_*) \tag{2.2}$$

where

$$R = \sqrt{(\varphi - \varphi_0)^2 + (\phi - \phi_0)^2}, \quad \omega = \operatorname{arctg}\frac{\phi - \phi_0}{\varphi - \varphi_0} \tag{2.3}$$

A and t_* are certain constants and n is a positive integer.

Differentiating each of relations (2.2) with respect to φ, we obtain

$$\sqrt{k_0}z_\varphi = -nAR^{n-1}\cos((n-1)\omega - t_*), \quad \theta_\varphi = nAR^{n-1}\sin((n-1)\omega - t_*) \tag{2.4}$$

Consequently, the point being investigated is regular when $n=1$, one isobar and one isocline passes through it, and the simultaneous vanishing of both components of the acceleration vector is excluded at this point. When $n \geq 2$, the point being investigated is a branch point at which, according to the more precise definition of a level line presented above, contact (not intersection) of n isobars and n isoclines occurs or, what is the same thing, an even number of $2n$ isobars and $2n$ isoclines emerge from this point as well as an even number $N=2(n-1)$ lines $F=0$ and the same number of lines $G=0$. We shall call the even number $N(N \geq 2)$ defined in this manner the index of the stationary point.

In the case of a circular bypass of a small neighbourhood of a branch point the signs of the longitudinal acceleration F strictly alternate in the lines $G=0$ and, in the same way, the signs of the function G strictly alternate in the lines $F=0$.

Substituting expressions (2.4) for z_φ and θ_φ into relations (1.3) when $N \geq 2$, we obtain the description of an infinite set of closed level lines $U=\text{const} \neq 0$ and $V=\text{const} \neq 0$. In particular, when $N=2$ in the physical plane in a small neighbourhood of a stationary point, we have two infinite families of ellipses $U=\text{const} \neq 0$ which touch at the stationary point of the line $U=0$ and two families of ellipses $V=\text{const} \neq 0$ which touch at the same point of the line $V=0$. We emphasize that the existence of closed level lines in the neighbourhood of a stationary point does not contradict the property of monotonicity. For example, for an appropriate choice of the bypass of the ellipse $U=\text{const} \neq 0$ when approaching the stationary point, the function V increases monotonically up to $+\infty$ and, after passing this point, it also decreases monotonically to $-\infty$.

2.2. An infinitely distant point (IDP)

The properties of subsonic flows at a considerable distance from a body and, in particular, the asymptotic forms have been investigated and discussed in a number of papers,[1,2,10,12–15] although many questions still remain open. Here, the problem solely consists of finding the number of zero lines of the components of the acceleration vector emerging from an IDP or the constraints imposed on this number. We note at once that it follows from the periodic solution that the required numbers are even.

The investigation of the structure of the level lines in the neighbourhood of an IDP depends very much on whether the circulation Γ and the value of the lift force Y, which is linearly related to it, are equal to zero. When $\Gamma \neq 0$, the dividing streamline $\psi=0$, which departs from the body into the flow, is a line of discontinuity of the potential φ and, at the same time in the (φ, ψ) plane of the potential, a given branch of the streamline $\psi=0$ will be a line of discontinuity of the gasdynamic parameters. The use of the Cauchy-Riemann type system (2.1) to analyse an IDP when $\Gamma \neq 0$ therefore requires additional justification. However, it is just when $\Gamma \neq 0$ that the flow in the neighbourhood of the IDP has been studied most completely. So, at a considerable distance from the body, the expressions for φ, q and θ, apart from small higher-order terms, have the form[1,2,13–15]

$$\begin{aligned} \varphi &= q_\infty x + \frac{\Gamma}{2\pi}\operatorname{arctg}\left(\frac{y}{x}\sqrt{1-M_\infty^2}\right) \\ -\frac{q-q_\infty}{y} &= \frac{q_\infty\theta}{x} = \frac{\Gamma\sqrt{1-M_\infty^2}}{2\pi(x^2+(1-M_\infty^2)y^2)} \end{aligned} \tag{2.5}$$

An analysis of these relations leads to the following conclusion: in the case of subsonic flow past bodies with a non-zero circulation, two isobars and two isoclines emerge and four lines $F=0$ and four lines $G=0$.

When $\Gamma=0$, relations (2.5) no longer hold. However, when $\Gamma=0$, the flow dividing line $\psi=0$ behind the body will no longer be a line of discontinuity of the potential. Consequently, when $\Gamma=Y=0$, there are no formal obstacles to the use of system (2.1) to analyse of the structure of level lines in the neighbourhood of an IDP. In this case, the periodic solution which is of interest is

$$\sqrt{k_\infty}(z-z_\infty) = AR^{-n}\cos(n\omega - t_*), \quad \theta = AR^{-n}\sin(n\omega - t_*) \tag{2.6}$$

where

$$R = \sqrt{\varphi^2+\phi^2}, \quad \omega = \operatorname{arctg}\frac{\phi}{\varphi}$$

A and t_* are certain constants and n is a positive intger.

The problem of constructing the asymptotic forms in the neighbourhood of an IDP involves the determination of all three parameters A, t_* and n. Here, only certain relations which the parameter n satisfies are of interest.

A value of $n \geq 2$ corresponds to a periodic solution, for which a greater number of level lines emerge from an IDP than when $\Gamma \neq 0$. An analysis of solution (2.6) shows that $2n$ isobars and $2n$ isoclines and $N_0 = 2(n+1)$ lines of zero values of the components of the acceleration vector emerge from an IDP and, here, in the case of a circular bypass of the IDP, the signs of the function F on the lines $G = 0$ strictly alternate as do the signs of the function G on the lines $F = 0$.

We see that, both when $\Gamma = 0$ and when $\Gamma \neq 0$, the number of zero-level lines of each of the components of the acceleration vector emerging from an IDP is greater than the number of isobars and isoclines emerging from the IDP while, in the case of a branch point located at an infinite distance from the body, the situation is exactly the opposite.

Finally, we will present the resultant relations for the numbers n and the indices N_0 in an IDP:

$$\Gamma \neq 0:\ n = 1,\quad N_0 = 4;\qquad \Gamma = 0:\ n \geq 2,\quad N_0 \geq 6$$

Remark. It has been noted above that, when $\Gamma \neq 0$, the flow in the neighbourhood of an IDP has been studied more thoroughly than when $\Gamma = 0$. This is due to the fact that, when $\Gamma \neq 0$, the first terms of the corresponding series can be successfully expressed in terms of Γ, which also leads to solution (2.5). When $\Gamma = 0$, the first important terms of the series are unknown in the general case with the exception of the special case of symmetric flow past a so-called "single vertex" body and, in particular, a convex body for which the sole constant required is expressed in terms of the area bounded by the streamline and its horizontal asymptote when the indicated streamline is moved upwards to infinity.[10] At the same time, the series describing the flow in the neighbourhood of an IDP when $\Gamma = 0$ are much simpler than when $\Gamma \neq 0$, which is obvious from the following relations for the potential φ[1,12–15]

$$\Gamma \neq 0:\ \varphi = q_\infty x + \frac{\Gamma}{2\pi}\operatorname{arctg}\left(\frac{y}{x}\sqrt{1-M_\infty^2}\right) + \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f_{nm}(\mu)\left(\frac{\ln r}{r}\right)^n \frac{1}{r^m}$$

$$\Gamma = 0:\ \varphi = q_\infty x + \sum_{m=1}^{\infty}\frac{g_m(\mu)}{r^m}$$

Here, x and y are Cartesian coordinates, $r = \sqrt{x^2+y^2}$, $\mathrm{u} = \operatorname{arctg}\frac{y}{x}$, $f_{nm}(\mu)$ and $g_m(\mu)$ are periodic functions to be determined.

It is obvious that the series corresponding to $\Gamma = 0$ or, more accurately, the first non-zero term with an index $m \geq 1$ can be used in the same way as solution (2.6) to determine the number of characteristic level lines emerging from an IDP.

3. Stationary points and zero-level lines of the components of the acceleration vector

It follows from the preceding section that no less than one stationary point exists in the subsonic flow past a body which is being considered. These are an IDP with an index $N_0 \geq 4$ and, possibly, $j (j \geq 1)$ branch points of the isobars and isoclines, each with an index $N_i \geq 2$ $(i = 1, \ldots, j)$ when $j = 0$, there are no branch points of the isobars and isoclines). The stationary points, the zero-level lines of the components of the acceleration vector emerging from them and the body in the flow are related by the following theorem.

Theorem 1. *Each zero-level line of any of the components of the acceleration vector which emerges from a stationary point, first, possesses the property that the other component of the acceleration vector outside the above-mentioned point has constant sign, and, second, this line reaches the surface of the body in the flow. In other words, the body surface is reached* $N_s = N_0 + N_1 + \ldots + N_j$ *lines* $G = 0$ *and the same number of lines* $F = 0$.

Proof. An even number N of zero-level lines of streamline curvature

$$G = V = 0 \tag{3.1}$$

emerge from a stationary point. In the circular bypassing of a stationary point, the lines (3.1), along which the function U increases during the motion from the stationary point, alternate with the lines along which U decreases. To be specific, we consider a line (3.1) along which U decreases. According to relations (1.3), the function U for this line simplifies to the expression $U = (kz_\varphi)^{-1}$. Consequently, the product kz_φ is equal to zero at the stationary point and increases during the motion from the stationary point along the line (3.1). Next, branch points of the streamlines and stagnation points are excluded in the subsonic flow being considered outside the body surface and, consequently, outside the body in the flow, the functions z and θ are bounded in modulus and the derivatives z_φ and θ_φ are therefore also bounded[1] in modulus outside the body (although the derivatives z_φ and θ_φ can reach values of $\pm\infty$ on the body). In other words, $0 < z_\varphi < \infty$, $0 < F < \infty$ along the line (3.1), which completes the proof of the first part of the theorem.

The monotonic increase in the product kz_φ along the line (3.1) excludes its self intersection and the inequality $z_\varphi < \infty$ precludes a situation in which the function U initially decreases from $+\infty$ to 0 and subsequently from 0 to $-\infty$ during the motion from the stationary point along the line (3.1) which, in its turn, precludes the line (3.1) from reaching any stationary point, including the initial stationary point. Finally, we will assume that the line (3.1) breaks off at a certain internal point of the flow w and consider a neighbourhood which may be as small as desired with its centre at the point w and lying as a whole in the flow domain. Simple analysis of the behaviour of the function G in this neighbourhood shows that an even number of points $G = 0$ exist in it, that is, apart from the point of intersection of the neighbourhood with the line (3.1), at least one further point $G = 0$ exists. Letting the radius of this neighbourhood tend to zero, we arrive at the refutation of the assumption which has been made.

Thus, the line (3.1) cannot self-intersect and form a closed loop, it cannot enter into another line which also includes the initial stationary point and, finally, it cannot break, inside the flow domain. Consequently, this line must arrive on the surface of the body in the flow. All that has been said can be automatically transferred to the other lines $G = V = 0$ and $F = U = 0$ which emerge from all the stationary points, which completes the proof.□

Remark.

1. We would hope that the properties of the lines $F = 0$ and $G = 0$ and the stationary points associated with them will enable the possibilities of topological methods to be extended in hydrodynamics[16] and, primarily, the method of level lines, the foundations of which were laid down in Ref. 7.
2. The number N_f (the number N_g) of zeros of the function F (of the function G) on the body in the flow and the number N_s introduced in the theorem are related by the inequalities $N_f \geq N_s$, $N_g \geq N_s$. These signs of the inequalities are due to the fact that points $F = 0$ ($G = 0$) can exist on a body which are points of a local extremum of the function F (of the function G). Moreover, points can exist on a body which are joined by the horseshoe-shaped lines $F = U = 0$ and $G = V = 0$ which do not contain the stationary points $F = G = 0$.

When investigating the flow past a number of known and newly constructed bodies, the inequality $N_g \geq N_s$ enables one to obtain the restriction on the over-all index N_s of the stationary points and (or) the quantity j of stationary points or to find the exact value of N_s. However, for this it is necessary to determine the number of zeros of the curvature of the streamlines on the body in the flow and, what is more important, the number of lines $G = V = 0$ which relate the body with the stationary points.

4. Zero levels of the curvature of streamlines on the body in the flow

We will now consider the determination of the number of zeros of the function G on the body using just the local properties of the flows in the neighbourhoods of the branch points of the dividing streamline and, in the following section, the discontinuity of the curvature.

We will consider a segment at of the dividing streamline which arrives on a smooth, kink-free segment of the generatrix of the body in the flow (Fig. 1). In an infinitesimal neighbourhood of point t which is a stagnation point, the flow is described by the equations for an incompressible fluid (the Mach number m in Eq. (1.1) is equal to zero) Apart from an unimportant constant factor, this flow is described by the following relations in a polar system of coordinates (R, ω) with its centre at the point t[17]

$$q = R, \quad \theta = -\omega + \pi \tag{4.1}$$

Here, R is the distance from point t and ω is the polar angle, the initial value of which, to be specific, is chosen such that $\omega = \pi$ corresponds to the segment at.

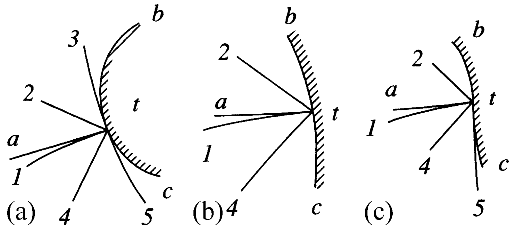

Fig. 1.

In other words, in a small neighbourhood of the stagnation point, semicircles correspond to the isobars and the straight lines correspond to the isoclines emerging from the point t. Next, omitting the calculations involved in obtaining the equations of the lines $F=0$ and $G=0$ from relations (4.1), we will present the main results.

Thus, in an infinitesimal neighbourhood of the point t, the segment at forms two right angles with the segments tb and tc. The bisectrices of these angles are the lines $F=0$, which are given with the numbers *2* and *4* in Fig. 1, where the curvature of the streamlines is positive (negative) in line *2* (line *4*). In turn, the lines $G=0$ touch the streamlines *at, tb* and *tc* at the point t. Regardless of the shape of the body in the neighbourhood of the stagnation point, the line $G=0$, which touches the dividing streamline, passes out into the flow domain. This line is given the number *1* in Fig. 1. The possibility of other lines $G=0$ emerging from point t depends on the curvature of the segments of the streamlines *tb* and *tc*. On account of this, we will consider three cases.

Case 1. The segments *tb* and *tc* convex towards the flow direction (Fig. 1a).The value of the function G in the segment *tb* is negative and, in line *2*, it is positive. Consequently, the line $G=0$, which is given the number *3*, emerges between lines *2* and *tb* into the flow domain along the tangent to *tb*. The line $G=0$, labelled with the number *5*, also emerges into the flow domain along the tangent to the line *tc*. Finally, three lines $G=0$, labelled with the numbers *1, 3* and *5* in Fig. 1a, emerge into the flow domain from the stagnation point t in the case being considered.

Case 2. The segments *tb* and *tc* are concave towards the flow direction (Fig. 1b). In this case, the function G has a single sign in the segment *tb* and in the line *2*. The same holds for the segments *tc* and *4*. Consequently, just a single line $G=0$ (line *1* in Fig. 1b) emerges into the flow domain.

Case 3. This is an intermediate case when the wall on one side of the stagnation point is convex and on the other side is concave. Without entering into a discussion of the possibility of the realizing such a situation when the branch point of the dividing streamline coincides with the point of inflection or, what is more, with the point of discontinuity in the sign of the curvature of the generatrix of the body, we merely note that, if such a situation occurs, two lines $G=0$, as, for example, lines *1* and *5* in Fig. 1c, emerge from the above-mentioned point into the flow domain in this case.

If the trailing branch point d from which the dividing streamline departs from the body is situated on a smooth, kink-free segment of the body, then everything that has been said above concerning the relative arrangement of the streamlines and the lines $G=0$ and $F=0$ in the neighbourhood of the leading branch point t holds for it. If the trailing branch point is a spinode with a non-zero angle, then, in this case, it will also be a stagnation point and, as analysis shows, the dependence of the number of lines $G=0$ emerging from it on the curvature of the adjacent segments of the body is the same as when there is no spinode. In this case, three lines $G=0$ emerge from the rear stagnation point if both of the adjacent segments are turned in a convex sense towards the flow direction, only one line $G=0$ if the two segments are concave with respect to the flow, and two lines $G=0$ if one of the segments is convex and the other is concave.

5. Singularity of the flow in the neighbourhood of a discontinuity point in the curvature of the wall of the body in the flow

We shall consider two cases.

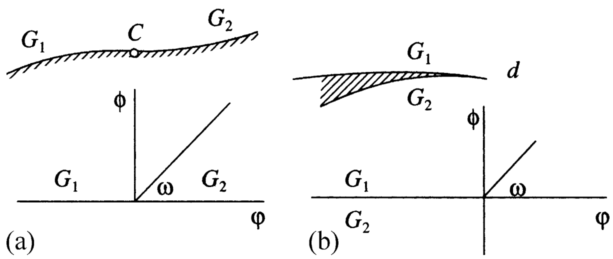

Fig. 2.

5.1. The case when the discontinuity point in the curvature is neither a branch point of a streamline nor a corner point of the wall

In this case, the gas-dynamic parameters q and θ are continuous in the neighbourhood of point c, and the velocity q is subsonic and is non-zero. However, the curvature of the wall on the two sides of the point c has different, but finite, values.

The parameters q and θ are continuous in the neighbourhood of point c and, therefore, a neighbourhood of point c, at each point of which q and θ differ by as small an amount as desired from q_c and θ_c, can be chosen, and we can therefore assume that the curvature of the streamlines in this neighbourhood is identical to the function G apart from a positive constant factor. The range of variation of the function G in the neighbourhood being considered is determined by the discontinuity in the curvature of the wall at point c, that is, it is a finite quantity. In order to find asymptotic expressions for the functions F and G in a polar system of coordinates, it is advisable to use system (1.2) as the initial system, assuming that the factor k is constant and equal to k_c in the neighbourhood of point c. With these assumptions, introducing the modified stream function $\phi = \sqrt{k_c}\psi$, we can transform system (1.2) into the following Cauchy-Riemann type system:

$$(\sqrt{k_c}U)_\varphi - V_\phi = 0, \quad (\sqrt{k_c}U)_\phi + V_\varphi = 0$$

Using relations (1.3), we can represent the latter system in the form

$$\left(\frac{\sqrt{k_c}z_\varphi}{k_c z_\varphi^2 + \theta_\varphi^2}\right)_\varphi - \left(\frac{\theta_\varphi}{k_c z_\varphi^2 + \theta_\varphi^2}\right)_\phi = 0, \quad \left(\frac{\sqrt{k_c}z_\varphi}{k_c z_\varphi^2 + \theta_\varphi^2}\right)_\phi + \left(\frac{\theta_\varphi}{k_c z_\varphi^2 + \theta_\varphi^2}\right)_\varphi = 0$$

In turn, this system is equivalent to a further Cauchy-Riemann type system

$$(\sqrt{k_c}z_\varphi)_\varphi + (\theta_\varphi)_\phi = 0, \quad (\sqrt{k_c}z_\varphi)_\phi - (\theta_\varphi)_\varphi = 0$$

which, if the expressions $F = z_\varphi q^3 \rho^{-1}$ and $G = \theta_\varphi q^3$ from Section 2 are taken into account, is equivalent to the system

$$(F\sqrt{1-M_c^2})_\varphi + G_\phi = 0, \quad (F\sqrt{1-M_c^2})_\phi - G_\varphi = 0$$

Finally, we will rewrite the resulting system in the more convenient polar system of coordinates associated with the point c,

$$(F\sqrt{1-M_c^2})_R + \frac{1}{R}G_\omega = 0, \quad \frac{1}{R}(F\sqrt{1-M_c^2})_\omega - G_R = 0 \tag{5.1}$$

The expressions for R and ω are given by formulae which are analogous to formulae (2.3) with the zero subscript replaced by the subscript c.

In order to investigate the singularities in the neighbourhood of point c, we will formulate and consider the following model problem.

Problem 1. A flow from left to right occurs on the wall in the neighbourhood of point c (Fig. 2a). Further to the left of point c, the transverse component of the acceleration vector is constant and equal to G_1 and, more to the right of this point $G = G_2$, $G_1 \neq G_2$. It is required to find the functions $F = F(R, \omega)$ and $G = G(R, \omega)$.

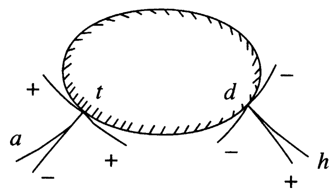

Fig. 3.

To solve this problem we will consider the upper half-plane of the (φ, ψ) plane shown in Fig. 2 in which $G = G_1$ when $\omega = \pi$ and $G = G_2$ when $\omega = 0$. In the case of the given boundary conditions, the solution of system (5.1) has the form

$$G = G_2 - \frac{G_2 - G_1}{\pi}\omega, \quad F\sqrt{1 - M_c^2} = \frac{G_2 - G_1}{\pi}\ln R$$

In other words, if, on moving along a streamline, the discontinuity $G_2 - G_1$ is positive (negative) at point c, then, on approaching point c from any direction, the longitudinal acceleration increases (decreases) logarithmically in an unbounded manner. In the physical plane in the neighbourhood of point c, the lines $F = \text{const}$ form a family of semi-ellipses, and the lines $G = \text{const}$ form a sheaf of lines where only when the quantities G_1 and G_2 have different signs will one of the lines of this sheaf be the line $G = 0$.

5.2. The case when the discontinuity in the curvature (but not in the angle of inclination of the velocity vector) occurs at the trailing branch point d

Model Problem 2 corresponds to this case.

Problem 2. Touching of the upper and the lower generatrices occurs at point d in Fig. 2b and the values of the curvature of the two generatrices are finite at point d, but not equal to one another. At point d, the streamlines corresponding to the upper and lower generatrices combine into the dividing streamline which departs to the right. In this case, the model problem is formulated as follows: $G = G_1$ when $\omega = \pi$ and $G = G_2$ when $\omega = -\pi$. The solution of system (5.1) takes the following form

$$G = \frac{G_1 + G_2}{2} - \frac{G_2 - G_1}{2\pi}\omega, \quad F\sqrt{1 - M_d^2} = \frac{G_2 - G_1}{2\pi}\ln R$$

It is obvious from the formulation of the problem that $G_1 > G_2$. Consequently, on approaching point d from any direction, the function F increases logarithmically in an unbounded manner. Finally, if G_1 and G_2 have the same sign, such as a negative sign, for example, as in Fig. 2b, then, in this case, not a single line $G = 0$ emerges from point d.

5.3. The case when point c in Fig. 2a is a point of inflection but not a point of discontinuity in the curvature

In this case, the problem of determining the number of lines $G = 0$ emerging from point c and the sign of the acceleration at this point becomes much more difficult and its solution in the general case may depend on the whole flow pattern. Practically all that has been said is also transferred to point d in Fig. 2b.

6. Flow past a body with a small number and with the minimum possible number of zeros of the function G

According to Theorem 1, there can be no less than four zeros of the function G and, at the same time, the existence of bodies in the case of four zeros of the function G is not obvious a priori. An example of such a body will be considered at the end of this section. For the present, we shall dwell on the flow past convex bodies.

Theorem 2. *In the case of a flow past smooth convex bodies* (Fig. 3) *with two branch points of the dividing streamline t and d, for non-zero values of the circulation Γ there are only two stationary points with a combined index $N_s = 6$.*

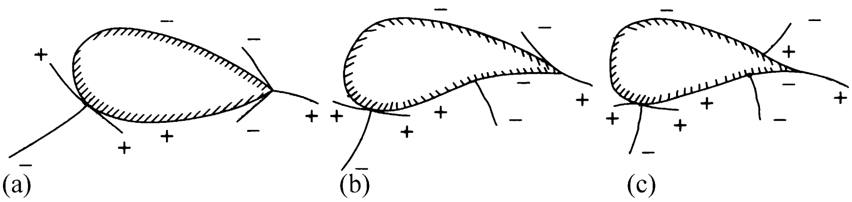

Fig. 4.

These are a IDP with the index $N_0 = 4$ and a single branch point of the isobars and isoclines with the index $N_1 = 2$. As the circulation tends to zero, the branch point moves away from the body and, in the case when the circulation is zero, the two stationary points merge into a single IDP with an index $N_0 = 6$.

Proof. In Fig. 3, *at* and *dh* are segments of the dividing streamline. Six lines $G = 0$ emerge from the surface of the body being considered into the flow domain: three each from points t and d. The plus and minus signs indicate the sign of the function F along the corresponding line $G = 0$. In the case of non-zero circulation, the solution in the neighbourhood of the IDP is described by relations (2.5) and, here, four lines $G = 0$ (the index $N_0 = 4$) arrive from the IDP onto the body. For two of them $F > 0$ and for the other two $F < 0$. Consequently, only one stationary point (the branch point of the isobars and isoclines) with an index $N_1 = 2$ exists at a finite distance from the body and a further two lines $G = 0$ arrive at the body from this point along one of which $F > 0$ and along the other $F < 0$.

Further, as was noted above, the index of the IDP $N_0 \geq 6$ in the case of zero circulation. However, only six lines $G = 0$ can arrive at the body being considered. Consequently, in the case being investigated of a flow with zero circulation $N_0 = 6$, there are no branch points at a finite distance from the body, which completes the proof.□

Corollary. *In the case of flow past a convex body with zero circulation, the six lines $G = 0$ which arrive at the body from infinity divide the flow domain into six subdomains in which the curvature of the streamlines has constant sign.*

Remarks.

1. Theorem 2 essentially generalizes the results in Refs. 9,10 in which it is proved that, for symmetrical flow past convex bodies, there are no branch points of the isobars and isoclines, and this also means that there are no stationary points beyond the IDP.
2. The results of the theorem also transfer to a number of other bodies from the surfaces of each of which six lines $G = 0$ also emerge into the flow domain. Examples of such bodies are shown in Fig. 4. Here, as in Fig. 3, the plus and minus signs on the lines $G = 0$ correspond to the sign of the function F, and the same signs on the segments of the body correspond to the sign of the function G. Note also that the well-known Zhukovskii profile refers to the class of bodies shown in Fig. 4c.

6.1. Flow past bodies with the minimum possible number of zeros of the function G

It is clear from the properties of an IDP considered above that no less than four lines $G = 0$ arrive from the IDP at the body in the flow and the index $N_0 = 4$ corresponds to flow with non-zero circulation. It has already been noted that the existence of bodies with a number of zeros of the function G equal to four is not obvious a priori. Nevertheless, it turns out that such a body can be constructed using the results of the two preceding sections.

Consider the following convex-concave body with a zero vertex angle at the trailing branch point (Fig. 5a and b). Here, *at* and *dh* are segments of the dividing streamline. The segment *cd* of the generatrix of the body is concave and the remaining part of the body is convex. The values of the curvature are finite on both sides of point c, they are non-zero and have different signs. Contact of the upper and lower generatrices, the values of the curvature of which are finite at point d, are different from one another and from zero and have the same sign, occurs at this point. To be specific, this sign has been taken as being negative in Fig. 5, which is not essential.

It follows from the properties of the branch points of the dividing streamline and the discontinuity of the curvature, which have been investigated above, that lines $G = 0$ only emerge from the surface of the body being considered into the flow domain from the stagnation point t and from point c of discontinuity of the curvature. The segments of the

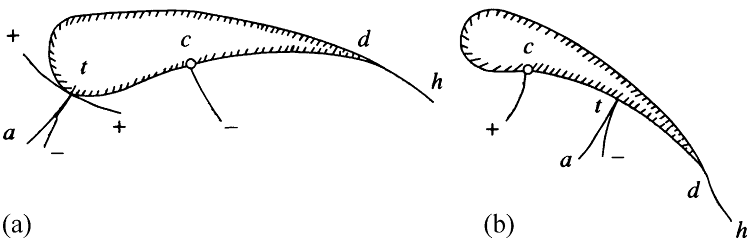

Fig. 5.

lines $G = 0$ associated with points t and c are shown in Fig. 5a and b where the plus and minus signs indicate the sign of the function F. An analysis of the above-mentioned lines $G = 0$ leads to the following assertions.

Theorem 3. *Continuous flow past a concave-convex body with non-identical branch points t and d of the dividing streamline* (Fig. 5) *is only possible when the leading branch point t is located on the convex part of the generatrix, as in* Fig. 5a, *and the flow is then characterized by non-zero values of the circulation and lift force and the existence of just a single stationary point, which is precisely a stationary IDP with* $N_0 = 4$. *As a consequence, the four lines* $G = 0$, *arriving at the body from infinity, divide the whole of the flow domain into four subdomains in which the curvature of the streamlines has a constant sign.*

Proof. Positioning of the point t in the concave segment, as in Fig. 5b as well as at the point c of discontinuity in the curvature leads to the fact that the number of zeros of the function G is equal to two, which contradicts the inequality $N_0 \geq 4$. Consequently, point t is located in a convex segment, as in Fig. 5a and the number of zeros of the function G is equal to four. Consequently, the values of the circulation and the lift force are non-zero, four lines $G = 0$ arrive at the body from the IDP (the index of the IDP $N_0 = 4$) and there are no stationary points other than the IDP.□

Remarks.

1. Attempts to prove Theorem 3 solely using an analysis of the isobars and isoclines do not lead to success, which yet again speaks of the new possibilities provided by an analysis of the zero-level lines of the components of the acceleration vector.
2. Theorem 3 leads to a number of paradoxes. We shall dwell on one of them. For this purpose, we consider the flow past a symmetrical Zhukovskii profile[18–20] within the framework of flows of an incompressible fluid. In the case of this profile, the number of zeros of the function G is equal to six, which can be seen from Fig. 4c. When the parameter responsible for the thickness of the Zhukovskii profile tends to zero, we obtain a profile which is as close as desired to an arc of a circle. By choosing the angle of attack for each of these profiles, a flow with zero values of the circulation and lift force and with an IDP index $N_0 = 6$ can be obtained for which the six lines $G = 0$ join the IDP and the profile. At the same time, a profile, which may be as close to an arc of a circle as desired, can be constructed on the basis of the concave-convex body shown in Fig. 5a. However, for any of these profiles, the number of zeros of the function G is equal to four, $\Gamma \neq 0$, $Y \neq 0$. We thereby arrive at a situation when fundamentally different flow patterns correspond to two profiles, which may be as close as desired. This and a number of other examples, the list of which can be continued, require individual consideration.

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References

1. Bers L. *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*. New York: Wiley; 1958.
2. Ovsyannikov LV. *Lectures on the Fundamentals of Gas Dynamics*. Moscow: Nauka; 1981.
3. Rylov AI. The properties of homogeneous systems of gas-dynamic equations for the components of the acceleration vector. *Sib Zh Industr Matematiki* 1998;**1**(2):169–74.

4. Rylov AI. Monotonicity properties of the solutions of first-order elliptic systems and their application to the equations of fluid mechanics. *Prikl Mat Mekh* 1995;**59**(5):758–66.
5. Mises R. *Mathematical Theory of Compressible Fluid Flow*. New York: Acad. Press; 1958.
6. Chernyi GG. *Gas Dynamics*. Moscow: Nauka; 1988.
7. Niko'skii AA, Taganov GI. The gas motion in a local supersonic zone and certain conditions for the disruption of a potential flow. *Prikl Mat Mekh* 1946;**10**(4):481–502.
8. Shifrin EG. Investigation of axially symmetric transonic flows using a special plane of a hodograph. *Prikl Mat Mekh* 1971;**35**(3):549–58.
9. Rylov AI. The monotonicity properties of certain plane vortex flows of an incompressible fluid and subsonic gas flows. *Prikl Mat Mekh* 1992;**56**(3):386–91.
10. Rylov AI. Asymptotic forms and the structure of level lines in subsonic plane potential flows. *Prikl Mat Mekh* 1999;**63**(3):403–14.
11. Rylov AI. Geometric properties of certain level lines in plane and axisymmetric gas flows. *Sib Zh Industr Matematiki* 2003;**6**(1):125–37.
12. Bateman H. Notes on a differential equation which occurs in the Two-dimensional motion of a compressible fluid and associated variational problem. *Proc Roy Soc London Ser A* 1929;**125**(799):598–618.
13. Ludford GSS. The behavior at infinity of the potential function Two-dimensional subsonic compressible flow. *J Math Phys* 1951;**30**(3):117–30.
14. Bers L. Existence and uniqueness of subsonic flow past a given profile. *Comm Pure Appl* 1954;**7**(3):441–504.
15. Finn R, Gilbarg D. Asymptotic behavior and uniqueness of plane subsonic flows. *Comm Pure Appl Math* 1957;**10**(1):23–63.
16. Arnold VI, Khesin BA. *Topological methods in hydrodynamics*. New York: Springer; 1997. p. 374.
17. Landau LD, Lifshitz EM. *Fluid Dynamics*. Oxford: Pergamon Press; 1987.
18. Kochin NE, Kibel' IA, Roze NV. *Theoretical Hydromechanics. Part 1*. Moscow: Gostekhizdat; 1955.
19. Vallander SV. *Lecture on Hydromechanics*. Leningrad: Izd. LGU; 1978.
20. Gorelov DN. *Methods of Solving Plane Boundary-Value Problems in Wing Theory*. Novosibirsk: Izd Sib Otd Ross Akad Nauk; 2000.

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